

CERN-TH/96-31

LP THE 96-09

ROME prep. 96/1131

ROM2F/96/06

## TWO-BODY NON-LEPTONIC DECAYS ON THE LATTICE

M. Ciuchini<sup>1,a</sup>, E. Franco<sup>2,b</sup>, G. Martinelli<sup>3</sup> and L. Silvestrini<sup>4</sup>

<sup>1</sup> Theory Division, CERN, 1211 Geneva 23, Switzerland.

<sup>2</sup> LP THE, Univ. de Paris XI, 91405 Orsay, France\*.

<sup>3</sup> Dip. di Fisica, Univ. “La Sapienza” and  
INFN, Sezione di Roma, P.le A. Moro, I-00185 Rome, Italy.

<sup>4</sup> Dip. di Fisica, Univ. di Roma “Tor Vergata” and INFN, Sezione di Roma II,  
Via della Ricerca Scientifica 1, I-00133 Rome, Italy.

### Abstract

We show that, under reasonable hypotheses, it is possible to study two-body non-leptonic weak decays in numerical simulations of lattice QCD. By assuming that final-state interactions are dominated by the nearby resonances and that the couplings of the resonances to the final particles are smooth functions of the external momenta, it is possible indeed to overcome the difficulties imposed by the Maiani-Testa no-go theorem and to extract the weak decay amplitudes, including their phases. Under the same assumptions, results can be obtained also for time-like form factors and quasi-elastic processes.

<sup>a</sup> On leave of absence from INFN, Sezione Sanità, V.le Regina Elena 299, 00161 Rome, Italy.

<sup>b</sup> On leave of absence from INFN, Sezione di Roma I, Dip. di Fisica, Università degli Studi “La Sapienza”, Rome, Italy.

\* Laboratoire associé au Centre National de la Recherche Scientifique.

# Introduction

Exclusive non-leptonic decays are a fundamental source of information on quark weak interactions and on the strong interaction dynamics. Unfortunately, a theoretical description of exclusive decays based on the fundamental theory is not possible yet. Over the years, several methods have been introduced to estimate the relevant matrix elements: vacuum saturation, bag models, quark models, QCD sum rules,  $1/N_c$  expansion, chiral Lagrangians, factorization, etc. None of these approaches is fully satisfactory. In the first three cases, the methods are in no way a systematic expansion of the fundamental theory in a small parameter. Thus the result of some calculation can be either correct or wrong, but it is impossible to reduce the uncertainty of the theoretical prediction by computing higher orders in the expansion. With QCD sum rules it is very difficult to improve the results in a systematic way because only few perturbative terms and matrix elements of higher dimensional operators can be estimated; in the  $1/N_c$  expansion nobody has succeeded in computing consistently the matrix elements of the effective Hamiltonian beyond the lowest order and chiral Lagrangians have a rather limited domain of application. It may well be true that factorization holds for weak decays of hadrons containing a heavy quark, although a real proof that this is the case is still lacking and moreover there are amplitudes that cannot be factorized.

The lattice approach has been used to obtain results based on first principles for a wide set of relevant physical quantities such as the hadron spectrum, the meson decay constants, the form factors entering in semileptonic and radiative decays, the kaon  $B$ -parameter  $B_K$ , etc. For exclusive non-leptonic decays, however, this method is no better than the above-mentioned approaches, because of the Maiani–Testa No-Go Theorem (MTNGT) and the activity in this field [1, 2] has completely stopped after the publication of ref. [3].

In this paper we show that, under quite reasonable physical hypotheses, it is possible to extract predictions for the relevant matrix elements in numerical simulations of lattice QCD, in spite of the MTNGT. Our proposal has a wide domain of applications, ranging from kaon decays,  $K \rightarrow \pi\pi$ , to charm and beauty decays,  $D \rightarrow K\pi$ ,  $B \rightarrow D\pi$ ,  $B \rightarrow \pi\pi$ , etc. If successful, it will also allow a check of several interesting theoretical ideas as the factorization of the amplitudes and the scaling laws that can be formulated in the Heavy Quark Effective Theory (HQET) when the mass of the quark becomes large. Under the same assumptions, it is possible to try a numerical calculation of the time-like form factors, e.g. those relevant

in  $e^+e^- \rightarrow \pi^+\pi^-$  or  $n\bar{n}$ , and of those entering in quasi-elastic electron–nucleon scattering processes, e.g.  $e^- + p \rightarrow e^- + \Delta + \pi$ .

The MTNGT states essentially the following:

- In the calculation of a two- (many-) body decay amplitude performed in the Euclidean space-time, which is the only possibility in Monte Carlo simulations, there is no distinction between *in*- and *out*-states. As a consequence, the matrix elements that one is able to extract are real numbers resulting from the average of the two cases, e.g.

$$\langle \pi\pi | \mathcal{H}_W | K \rangle = \frac{1}{2} \left( {}_{in} \langle \pi\pi | \mathcal{H}_W | K \rangle + {}_{out} \langle \pi\pi | \mathcal{H}_W | K \rangle \right). \quad (1)$$

Any trace of the phase due to final-state interactions is then lost. This jeopardizes the possibility of any realistic prediction for the matrix elements. For example, we know from the measured  $A_{1/2}$  and  $A_{3/2}$  amplitudes in  $D \rightarrow K\pi$  decays that there is a phase difference of about  $80^\circ$ .

- Matrix elements are extracted on the lattice by studying the time behaviour of appropriate correlation functions at large time distances. Maiani and Testa showed that what can be really isolated in this limit are the off-shell form factors corresponding to the final particles at rest, e.g.  $\langle \pi(\vec{p}_\pi = 0) \pi(\vec{p}_\pi = 0) | \mathcal{H}_W | K \rangle$ . For kaon decays, we can use the chiral theory to extrapolate the form factor to the physical point<sup>1</sup>. This is certainly not the case for  $D$ - and  $B$ -meson decays. In the latter case it is not possible to obtain a realistic prediction for the matrix element.

We will show that both the difficulties raised by Maiani and Testa can be overcome under the hypothesis that final-state interactions are dominated by nearby resonances, the couplings of which to the final-state particles satisfy some smoothness condition<sup>2</sup>. By varying the spatial momentum of the initial and final hadrons, it is then possible to extract the physical matrix elements, including the phase due to the strong-interaction rescattering of the final states. This allows also the calculation of the relevant parameters of the resonances.

Two observations are necessary at this point.

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<sup>1</sup> In the chiral limit final-state interactions are negligible and the problem discussed before is solved. This is thus the only situation in which the matrix element can be evaluated.

<sup>2</sup> A different approach, based on the study of the two-particle energy spectrum in a periodic box can be found in ref. [4]

The assumption that final-state interactions are dominated by resonances is not new and has been successfully applied to phenomenological studies of charmed meson decays, see for example ref. [5] (together with the factorization hypothesis that we need not assume).

At this stage, we do not know if the procedure that we are proposing can lead to useful results in practice. It remains to be seen whether, with a reasonable size of the lattice and number of gauge field configurations, it is possible to extract the matrix elements with a satisfactory accuracy. A feasibility study is currently under way on the APE machine.

## The three-point Euclidean correlation function

Following ref. [3], we first examine the Euclidean three-point function  $G(t_1, t_2, \vec{q}, -\vec{q})$

$$G(t_1, t_2, \vec{q}, -\vec{q}) \equiv \langle 0 | T [\Pi_{\vec{q}}(t_1) \Pi_{-\vec{q}}(t_2) H(0)] | 0 \rangle = \langle 0 | \Pi_{\vec{q}}(t_1) \Pi_{-\vec{q}}(t_2) H(0) | 0 \rangle, \quad (2)$$

when  $t_1 > t_2 > 0$ . In eq. (2),  $\Pi_{\vec{q}}(t)$  is an interpolating field of the final-state particle (denoted as “pion” in the following) with a fixed spatial momentum

$$\Pi_{\vec{q}}(t) = \int d^3x e^{-i\vec{q}\cdot\vec{x}} \Pi(\vec{x}, t); \quad (3)$$

$H(0) = H(\vec{x} = 0, t = 0)$  is any local operator that couples to the two pions in the final state;  $T[\dots]$  represents the  $T$ -product of the fields and the vacuum expectation value corresponds, in a numerical simulation, to the average over the gauge field configurations.

When  $t_1 \rightarrow \infty$

$$G(t_1, t_2, \vec{q}, -\vec{q}) \rightarrow \sum_n \langle 0 | \Pi_{\vec{q}}(t_1) | n \rangle \langle n | \Pi_{-\vec{q}}(t_2) H(0) | 0 \rangle \sim \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{-E_{\vec{q}} t_1} G_3(t_2), \quad (4)$$

where

$$E_{\vec{q}} = \sqrt{M_\pi^2 + \vec{q}^2}, \quad \sqrt{Z_\Pi} = \langle 0 | \Pi(0) | \vec{q} \rangle \quad (5)$$

and

$$G_3(t) = \langle \vec{q} | \Pi_{-\vec{q}}(t) H(0) | 0 \rangle \quad (6)$$

for  $t > 0$ .

Inserting a complete set of *out*-states, we can write

$$\langle \vec{q} | \Pi_{-\vec{q}}(t) H(0) | 0 \rangle = \sum_n (2\pi)^3 \delta^3(P_n) \langle \vec{q} | e^{\hat{H}t} \Pi(0) e^{-\hat{H}t} | n \rangle_{out} \langle n | H(0) | 0 \rangle =$$

$$\begin{aligned}
& \sum_n (2\pi)^3 \delta^3(P_n) \langle \vec{q} | \Pi(0) | n \rangle_{out\ out} \langle n | H(0) | 0 \rangle e^{-(E_n - E_{\vec{q}})t} = \\
& \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{-E_{\vec{q}}t} \times {}_{out} \langle \vec{q}, -\vec{q} | H(0) | 0 \rangle + \\
& \left[ \sum_n (2\pi)^3 \delta^3(P_n) \langle \vec{q} | \Pi(0) | n \rangle_{out\ out} \langle n | H(0) | 0 \rangle e^{-(E_n - E_{\vec{q}})t} \right]_{\text{connected}}, \tag{7}
\end{aligned}$$

where the term proportional to  ${}_{out} \langle \vec{q}, -\vec{q} | H(0) | 0 \rangle$  on the r.h.s. is the disconnected contribution.

We now make the following assumptions:

1. The only possible eigenstates of the Hamiltonian are  $n$ -pion states.
2. The interaction between pions is dominated by a narrow resonance (denoted as  $\sigma$  in the following) exchanged in the  $s$ -channel.
3. The coupling of  $\sigma$  to the pions is a smooth function of the external momenta.

Under hypotheses 1. and 2., the only possible intermediate states are two-pion states. We thus obtain

$$\begin{aligned}
G_3(t) &= \sum_n (2\pi)^3 \delta^3(P_n) \langle \vec{q} | \Pi(0) | n \rangle_{out\ out} \langle n | H(0) | 0 \rangle e^{-(E_n - E_{\vec{q}})t} = \\
& \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{-E_{\vec{q}}t} \times \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} + \\
& \left[ \sum_n (2\pi)^3 \delta^3(P_n) \langle \vec{q} | \Pi(0) | n \rangle_{out\ out} \langle n | H(0) | 0 \rangle e^{-(E_n - E_{\vec{q}})t} \right]_{\text{connected}}. \tag{8}
\end{aligned}$$

In eq. (8)  $\sqrt{s_q} = 2E_{\vec{q}}$ ;  $|n\rangle_{out} = |\vec{k}, -\vec{k}\rangle_{out}$ ;  $\sqrt{s} = E = 2E_{\vec{k}}$ ; the sum over the intermediate states is given by<sup>3</sup>

$$\begin{aligned}
\sum_n (2\pi)^3 \delta^3(P_n) &= \int \frac{d^3k_1}{(2\pi)^3 2E_{\vec{k}_1}} \frac{d^3k_2}{(2\pi)^3 2E_{\vec{k}_2}} (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) = \\
& \int \frac{dE}{2\pi} \int \frac{d^3k_1}{(2\pi)^3 2E_{\vec{k}_1}} \frac{d^3k_2}{(2\pi)^3 2E_{\vec{k}_2}} (2\pi)^4 \delta^3(\vec{k}_1 + \vec{k}_2) \delta(E_{\vec{k}_1} + E_{\vec{k}_2} - E) \tag{9}
\end{aligned}$$

and we have used

$${}_{out} \langle \vec{q}, -\vec{q} | H(0) | 0 \rangle = \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} \Big|_{s_q=p_H^2}. \tag{10}$$

We now define

$$\begin{aligned}
\langle \vec{q} | \Pi(0) | \vec{k}, -\vec{k} \rangle_{out} &= \left[ \frac{2\sqrt{Z_\Pi}}{(E + 2E_{\vec{q}} - i\epsilon)(-E + 2E_{\vec{q}} - i\epsilon)} \frac{|V(s)|^2}{M_\sigma^2 - s - iX(s)} \right]^* \\
&= \frac{2\sqrt{Z_\Pi}}{(E + 2E_{\vec{q}} + i\epsilon)(-E + 2E_{\vec{q}} + i\epsilon)} \frac{|V(s)|^2}{M_\sigma^2 - s + iX(s)} \tag{11}
\end{aligned}$$

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<sup>3</sup>A further factor 1/2 in the case of identical particles is understood.

In eq. (11) we have introduced the  $\sigma$ - $\pi$ - $\pi$  coupling  $V(s)$ . As stated before, we assume here that the coupling is such a smooth function of the external momenta that we can use the same “physical” coupling also for the off-shell pion, which is annihilated by  $\Pi(0)$  in the matrix element  $\langle \vec{q} | \Pi(0) | \vec{k}, -\vec{k} \rangle_{out}$ . In general we should write

$$\begin{aligned} {}_{out} \langle \vec{k}, -\vec{k} | \Pi(0) | \vec{q} \rangle &= \sqrt{Z_\Pi} \times \left[ \frac{2}{(E + 2E_{\vec{q}} - i\epsilon)(-E + 2E_{\vec{q}} - i\epsilon)} \right] \times \\ &\frac{|V(s)|^2}{M_\sigma^2 - s - iX(s)} \times \mathcal{F} \left( 1 - \frac{E}{2E_{\vec{q}}} \right), \end{aligned} \quad (12)$$

with the condition that the modulating factor  $\mathcal{F}$  of the off-shellness  $1 - E/2E_{\vec{q}}$  satisfies the condition  $\mathcal{F}(0) = 1$  (see below). The factor in parenthesis is (up to a factor  $1/2E_{\vec{q}}$ ) the propagator of two non-interacting pions [6].

With the definition in eq. (11), the quantity in square brackets satisfies

$$\begin{aligned} \lim_{E \rightarrow \sqrt{s} \rightarrow 2E_{\vec{q}}} (-p^2 + M_\pi^2) &\left[ \frac{2\sqrt{Z_\Pi}}{(E + 2E_{\vec{q}} - i\epsilon)(-E + 2E_{\vec{q}} - i\epsilon)} \frac{|V(s)|^2}{M_\sigma^2 - s - iX(s)} \right] \rightarrow \\ \sqrt{Z_\Pi} \frac{|V(s_q)|^2}{M_\sigma^2 - s_q - iX(s_q)} &= \sqrt{Z_\Pi} \mathcal{A}(\pi(\vec{q}) + \pi(-\vec{q}) \rightarrow n = \pi(\vec{k}) + \pi(-\vec{k}))|_{|\vec{k}|=|\vec{q}|}, \end{aligned} \quad (13)$$

which is the usual LSZ-reduction formula for on-shell particles, cf. eq. (15) of ref. [3]. In eq. (13),  $p^2 = E^2 + M_\pi^2 - 2E E_{\vec{q}}$  is the squared four-momentum of the off-shell pion. Combining eqs. (8) and (11) and using the identity

$$\frac{1}{E - 2E_{\vec{q}} - i\epsilon} = \mathcal{P} \left[ \frac{1}{E - 2E_{\vec{q}}} \right] + i\pi\delta(E - 2E_{\vec{q}}), \quad (14)$$

we obtain

$$\begin{aligned} G_3(t) &= \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{E_{\vec{q}}t} \times \left[ \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} e^{-2E_{\vec{q}}t} \right. \\ &\quad \left. + (4E_{\vec{q}}) \int \frac{dE}{\pi} \frac{g(s)}{(E + 2E_{\vec{q}} + i\epsilon)(-E + 2E_{\vec{q}} + i\epsilon)} \frac{X(s)}{(M_\sigma^2 - s)^2 + X^2(s)} e^{-Et} \right] \\ &= \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{E_{\vec{q}}t} \times \left\{ \left( \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} - \frac{ig(s_q)X(s_q)}{(M_\sigma^2 - s_q)^2 + X^2(s_q)} \right) e^{-2E_{\vec{q}}t} \right. \\ &\quad \left. + (4E_{\vec{q}}) \mathcal{P} \left[ \frac{1}{\pi} \int_{2M_\pi}^{+\infty} dE e^{-Et} \frac{g(s)\rho(s)}{4E_{\vec{q}}^2 - E^2} \right] \right\}, \end{aligned} \quad (15)$$

where, in the integrand of the r.h.s., we have denoted as  $E^2$  or  $s$  the same quantity,  $s = E^2$ . This will facilitate the comparison of the above formulae with the more general case of a two-pion final state with total momentum different from zero, cf. eqs. (40) and (41) below. In

eq. (15) we have used the relation  $X(s) = M_\sigma \Gamma(s)$ , where  $\Gamma(s)$  is the “width” of the  $\sigma$ -meson defined as

$$\begin{aligned}\Gamma(s) &= \frac{1}{2M_\sigma} \left[ \int \frac{d^3 k_1}{(2\pi)^3 2E_{\vec{k}_1}} \frac{d^3 k_2}{(2\pi)^3 2E_{\vec{k}_2}} (2\pi)^4 \delta^3(\vec{k}_1 + \vec{k}_2) \delta(E_{\vec{k}_1} + E_{\vec{k}_2} - \sqrt{s}) |V(s)|^2 \right] \\ &= \frac{|V(s)|^2}{8\pi M_\sigma} \sqrt{\frac{(s - 4M_\pi^2)}{4s}} \theta(s - 4M_\pi^2).\end{aligned}\quad (16)$$

The spectral function  $\rho(E^2)$  is given by

$$\rho(E^2) = \frac{X(E^2)}{(M_\sigma^2 - E^2)^2 + X(E^2)^2},$$

which becomes

$$\rho(E^2) = \frac{1}{2M_\sigma} \frac{\Gamma(E^2)/2}{(M_\sigma - E)^2 + (\Gamma(E^2)/2)^2} \quad (17)$$

for a narrow resonance. We also have

$$\rho(E^2) \rightarrow \frac{\pi}{2M_\sigma} \delta(M_\sigma - E) \quad (18)$$

as  $\Gamma \rightarrow 0$ .

We then establish the relation

$$\begin{aligned}& \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} - \frac{ig(s_q)X(s_q)}{(M_\sigma^2 - s_q)^2 + X^2(s_q)} = \\ & \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} + \frac{1}{2} \left( \frac{g(s_q)}{M_\sigma^2 - s_q + iX(s_q)} - \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} \right) = \\ & {}_{out}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle + \frac{1}{2} \left( {}_{in}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle - {}_{out}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle \right),\end{aligned}\quad (19)$$

in agreement with eq. (21) of ref. [3]. The last term in eq. (15) corresponds to the last term of eq. (23) of ref. [3]. The definitions of  ${}_{out}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle$  and  ${}_{in}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle$  are in agreement with the standard relation between the strong rescattering phase and the parameters of the resonance  $\sigma$

$${}_{out}\langle \vec{q}, -\vec{q} | \vec{q}, -\vec{q} \rangle_{in} = e^{2i\delta(s)} = \frac{M_\sigma^2 - s + iX(s)}{M_\sigma^2 - s - iX(s)}, \quad (20)$$

where the phase  $\delta(s)$  is defined up to an ambiguity of  $\pi \equiv 180^\circ$ . We choose  $0 \leq \delta(s) \leq \pi$ .

In the case of non-interacting pions,  ${}_{out}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle = {}_{in}\langle \vec{q}, -\vec{q} | H(0) | 0 \rangle = \sqrt{Z_\sigma}$ , and one finds

$$G(t_1, t_2, \vec{q}, -\vec{q}) = \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} \sqrt{Z_\sigma}. \quad (21)$$

Equation (21) has a very simple interpretation. The two non-interacting particles are created in the origin  $t = 0$  with amplitude  $\sqrt{Z_\sigma}$  and propagate from the origin to  $t_2$ ; from  $t_2$  to  $t_1$  only one particle state propagates.

In the limit of a zero-width resonance ( $X(s) \rightarrow 0$ ) we obtain

$$G(t_1, t_2, \vec{q}, -\vec{q}) = \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 \frac{e^{-E_{\vec{q}}(t_1+t_2)}}{M_\sigma^2 - s_q} \left[ g(s_q) - g(M_\sigma^2) \left( \frac{2E_{\vec{q}}}{M_\sigma} \right) e^{-(M_\sigma - 2E_{\vec{q}})t_2} \right], \quad (22)$$

which is in agreement with the result, obtained for a narrow resonance, given in eq. (3.3) of ref. [6]. We notice that eq. (3.3) of ref. [6] is really valid only at small time-distances. According to the MTNGT instead, for any non-zero value of the width, the correlation function at large time-distances is always dominated by the state with the two pions at rest.

For further use, it is convenient to introduce the quantity  $R^H(t_2, \vec{q})$  defined from the relation

$$G(t_1, t_2, \vec{q}, -\vec{q}) = \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} R^H(t_2, \vec{q}), \quad (23)$$

where the label  $H$  in  $R^H(t_2, \vec{q})$  denotes the local operator used to create the two pions.

It is straightforward at this point to derive the expression that enters in a two-body non-leptonic decay of a meson (which we will denote as  $D$  in the following). The starting point is the four-point correlation function

$$\begin{aligned} G(t_1, t_2, t_D, \vec{q}, -\vec{q}, \vec{p}_D = 0) &= \lim_{t_D \rightarrow -\infty, t_1 \rightarrow +\infty} \langle \Pi_{\vec{q}}(t_1) \Pi_{-\vec{q}}(t_2) \mathcal{H}_W(0) D_{\vec{p}_D=0}^\dagger(t_D) \rangle \\ &= \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} R^{\mathcal{H}_W}(t_2, \vec{q}) \frac{\sqrt{Z_D}}{2M_D} e^{-M_D|t_D|}, \end{aligned} \quad (24)$$

with

$$\begin{aligned} R^{\mathcal{H}_W}(t_2, \vec{q}) &= \frac{g^W(s_q)}{M_\sigma^2 - s_q - iX(s_q)} + \frac{1}{2} \left( \frac{g^W(s_q)}{M_\sigma^2 - s_q + iX(s_q)} - \frac{g^W(s_q)}{M_\sigma^2 - s_q - iX(s_q)} \right) + \\ &(4E_{\vec{q}}) \mathcal{P} \left[ \int_{2M_\pi}^{+\infty} \frac{dE}{\pi} \frac{e^{-(E-2E_{\vec{q}})t_2}}{4E_{\vec{q}}^2 - E^2} \frac{1}{2i} \left( \frac{g^W(s)}{M_\sigma^2 - s - iX(s)} - \frac{g^W(s)}{M_\sigma^2 - s + iX(s)} \right) \right], \end{aligned} \quad (25)$$

and

$${}_{out} \langle \vec{k}, -\vec{k} | \mathcal{H}_W(0) | D(\vec{p}_D = 0) \rangle = \frac{g^W(s)}{M_\sigma^2 - s - iX(s)}, \quad (26)$$

where  $\mathcal{H}_W$  is the weak Hamiltonian. We have not considered complications coming from the presence of (axial-)vector particles and different flavours. The formulae reported in this section can easily be extended to these cases.



# The Watson theorem and physical interpretation of the results

Let us start from

$$\begin{aligned}
G(t_1, t_2, \vec{q}, -\vec{q}) &= \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} \times \left\{ \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} + \right. \\
&\frac{1}{2} \left( \frac{g(s_q)}{M_\sigma^2 - s_q + iX(s_q)} - \frac{g(s_q)}{M_\sigma^2 - s_q - iX(s_q)} \right) + \\
&\left. (4E_{\vec{q}}) \mathcal{P} \left[ \int_{2M_\pi}^{+\infty} \frac{dE}{\pi} e^{-(E-2E_{\vec{q}})t_2} \frac{1}{4E_{\vec{q}}^2 - E^2} \frac{1}{2i} \left( \frac{g(s)}{M_\sigma^2 - s - iX(s)} - \frac{g(s)}{M_\sigma^2 - s + iX(s)} \right) \right] \right\},
\end{aligned} \tag{27}$$

and introduce the following relation

$${}_{out} \langle \vec{k}, -\vec{k} | H(0) | 0 \rangle = \frac{g(s)}{M_\sigma^2 - s - iX(s)} = e^{i\delta(s)} A(s). \tag{28}$$

The Watson theorem ensures us that (in the absence of CP violation)  $A(s)$  is a real quantity [7]. It follows that

$$g(s) = \sqrt{(M_\sigma^2 - s + iX(s))(M_\sigma^2 - s - iX(s))} \times A(s) \tag{29}$$

is also a real quantity. Equation (20) implies

$${}_{in} \langle \vec{k}, -\vec{k} | H(0) | 0 \rangle = \frac{g(s)}{M_\sigma^2 - s + iX(s)} = e^{-i\delta(s)} A(s). \tag{30}$$

We may then write

$$G(t_1, t_2, \vec{q}, -\vec{q}) = \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} \times \tag{31}$$

$$\left\{ A(s_q) \cos \delta(s_q) + (4E_{\vec{q}}) \mathcal{P} \left[ \int_{2M_\pi}^{+\infty} \frac{dE}{\pi} e^{-(E-2E_{\vec{q}})t_2} \frac{1}{4E_{\vec{q}}^2 - E^2} A(s) \sin \delta(s) \right] \right\},$$

with

$$\cos \delta(s) = \frac{M_\sigma^2 - s}{\sqrt{(M_\sigma^2 - s)^2 + X(s)^2}} \tag{32}$$

$$\sin \delta(s) = \frac{X(s)}{\sqrt{(M_\sigma^2 - s)^2 + X(s)^2}}. \tag{33}$$

Notice that at threshold  $\delta(s)|_{s=4M_\pi^2} = 0$ , cf. eq. (16).

Following ref. [3], we can compute the leading behaviour of  $G(t_1, t_2, \vec{q} = 0, -\vec{q} = 0)$  for large  $t_2$  and express the result in terms of the parameters of the resonance. We obtain

$$G(t_1, t_2, \vec{q} = 0, -\vec{q} = 0) = \frac{Z_\Pi}{4M_\pi^2} e^{-M_\pi(t_1+t_2)} A(4M_\pi^2) \left[ 1 - a \sqrt{\frac{M_\pi}{\pi t_2}} \right], \quad (34)$$

where the scattering length  $a$  is given by

$$a = \frac{|V(4M_\pi^2)|^2}{16\pi M_\pi} \frac{1}{M_\sigma^2 - 4M_\pi^2}, \quad (35)$$

in agreement with eq. (10) of ref. [3]<sup>4</sup>.

## Applications to lattice calculations

For  $D$ -decays eq. (31) becomes

$$\begin{aligned} G(t_1, t_2, t_D, \vec{q}, -\vec{q}, \vec{p}_D = 0) &= \left( \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} \right)^2 e^{-E_{\vec{q}}(t_1+t_2)} \times \\ &\left\{ A^W(s_q) \cos \delta(s_q) + (4E_{\vec{q}}) \mathcal{P} \left[ \int_{2M_\pi}^{+\infty} \frac{dE}{\pi} e^{-(E-2E_{\vec{q}})t_2} \frac{1}{4E_{\vec{q}}^2 - E^2} A^W(s) \sin \delta(s) \right] \right\} \\ &\times \frac{\sqrt{Z_D}}{2M_D} e^{-M_D|t_D|}, \end{aligned} \quad (36)$$

where  $g^W(s) = \sqrt{(M_\sigma^2 - s + iX(s))(M_\sigma^2 - s - iX(s))} \times A^W(s)$ .

For numerical applications, it is convenient to consider the amputated correlation function given by the ratio

$$R^{\mathcal{H}W}(t_2, \vec{q}) = \frac{G(t_1, t_2, t_D, \vec{q}, -\vec{q}, \vec{p}_D = 0)}{S_\Pi(t_1, E_{\vec{q}}) S_\Pi(t_2, E_{\vec{q}}) S_D(t_D, M_D)}, \quad (37)$$

where

$$S_\Pi(t_1, E_{\vec{q}}) = \frac{\sqrt{Z_\Pi}}{2E_{\vec{q}}} e^{-E_{\vec{q}}t_1}, \quad (38)$$

and similarly for the other meson propagators;

$$R^{\mathcal{H}W}(t_2, \vec{q}) = \left\{ A^W(s_q) \cos \delta(s_q) + (4E_{\vec{q}}) \mathcal{P} \left[ \int_{2M_\pi}^{+\infty} \frac{dE}{\pi} e^{-(E-2E_{\vec{q}})t_2} \frac{1}{4E_{\vec{q}}^2 - E^2} A^W(s) \sin \delta(s) \right] \right\}. \quad (39)$$

Until now, for simplicity, we had chosen  $\vec{p}_D = 0$ ; the generalization of the above formulae to  $\vec{p}_D \neq 0$ , corresponding to  $G(t_1, t_2, t_D, \vec{q}_1, -\vec{q}_2, \vec{p}_D = \vec{q}_1 - \vec{q}_2)$ , is straightforward:

$$R^{\mathcal{H}W}(t_2, \vec{q}_1, -\vec{q}_2) = \frac{G(t_1, t_2, t_D, \vec{q}_1, -\vec{q}_2, \vec{p}_D = \vec{q}_1 - \vec{q}_2)}{S_\Pi(t_1, E_{\vec{q}_1}) S_\Pi(t_2, E_{\vec{q}_2}) S_D(t_D, E_{\vec{p}_D})} =$$

---

<sup>4</sup> There is a difference of a factor of two, due to the fact that in our case the particles are distinguishable.

$$\left\{ A^W(s_T) \cos \delta(s_T) + (2E_T) \mathcal{P} \left[ \int_{E_{min}}^{+\infty} \frac{dE}{\pi} e^{-(E-E_T)t_2} \frac{1}{E_T^2 - E^2} A^W(s) \sin \delta(s) \right] \right\}, \quad (40)$$

where

$$E_T = E_{\vec{q}_1} + E_{\vec{q}_2}; \quad E_{min} = \sqrt{4M_\pi^2 + (\vec{q}_1 - \vec{q}_2)^2}; \quad s_T = E_T^2 - (\vec{q}_1 - \vec{q}_2)^2 \quad (41)$$

and  $s = E^2 - (\vec{q}_1 - \vec{q}_2)^2$ .

We now explain the strategy to extract the physical information, i.e. the matrix element of  $\mathcal{H}_W$ , including the phase, from  $R^{\mathcal{H}_W}(t_2, \vec{q})$ . The extension to  $R^{\mathcal{H}_W}(t_2, \vec{q}_1, -\vec{q}_2)$  is straightforward.

Let us imagine to make a calculation on a lattice of volume  $L^3 \times T$ . The lattice version of eq. (15) is given by<sup>5</sup>:

$$R^{\mathcal{H}_W}(t_2, \vec{q}) = A(s_q) \cos \delta(s_q) + \left( \frac{4E_{\vec{q}}}{\pi} \right) \sum'_{E_i} \left[ \Delta E_i e^{-(E_i - 2E_{\vec{q}})t_2} \frac{1}{4E_{\vec{q}}^2 - E_i^2} A(s) \sin \delta(s) \right], \quad (42)$$

where all the quantities are given in units of the lattice spacing,

$$E_i = \sqrt{s} = 2E_{\vec{k}} \quad \text{with} \quad \vec{k} \equiv \frac{2\pi}{L}(n_x, n_y, n_z); \quad (43)$$

$s_q$  has been defined before;  $n_{x,y,z} = 0, 1, \dots, L-1$  and  $\sum'_{E_i}$  denotes the sum over all the values of the energy corresponding to the momenta  $\vec{k}$  allowed by the discretization of the space-time on a finite volume, excluding those corresponding to  $E_i = 2E_{\vec{q}}$ . Different combinations of momenta corresponding to the same energy should be included only once in the sum appearing in eq. (42), since the factors multiplying  $\sum'_{E_i}$  already account for the phase space integration, at fixed total energy. Thus, for example,  $|k| = 2\pi/L$  corresponds to six possibilities<sup>6</sup>,  $(\pm 1, 0, 0)$ ,  $(0, \pm 1, 0)$ , and  $(0, 0, \pm 1)$ , but has to be counted as a single term in the sum.  $\Delta E_i = E_{i+1} - E_i$  is the difference of the nearest successive allowed values of  $E_i$  ( $E_0 = 2M_\pi$ ,  $E_1 = 2\sqrt{M_\pi^2 + (2\pi/La)^2}$ ,  $E_2 = 2\sqrt{M_\pi^2 + 2(2\pi/La)^2}$ , etc.). One can show that the expression in eq. (42) tends to the corresponding continuum one in eq. (39) as  $L \rightarrow \infty$ .

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<sup>5</sup> We assume here that the discretized version of the two-body phase space is a good approximation of the continuum one. In this case we can change the integrals into sums over discrete values of momenta or energies without problems. Further technical complications arising from the different density of states in the continuum and on the lattices currently used in numerical simulations will be discussed elsewhere.

<sup>6</sup> In the following,  $\vec{k} \equiv 2\pi/L(n_x, n_y, n_z)$  will be simply denoted by  $(1, 0, 0)$ .

By varying the external momenta we can study

$$\tan \delta(s_q) = \frac{A_s}{A_c} \quad \text{and} \quad |A(s_q)| = \sqrt{A_c^2 + A_s^2} \quad (44)$$

as a function of the centre-of-mass energy, where  $A_c = A(s_q) \cos \delta(s_q)$  and  $A_s = A(s_q) \sin \delta(s_q)$ . From the behaviour of  $\tan \delta(s_q)$  as a function of  $s_q$ , we can reconstruct the mass and width of the resonance and extrapolate the amplitude to the physical point.

$A_c$  and  $A_s$  can be found by studying the dependence of  $R^{\mathcal{H}w}(t_2, \vec{q})$  on  $t_2$ , at fixed  $\vec{q}$ , for  $t_1 \gg t_2 \gg 0$ .

Several observations are important here:

- The range of values of  $s_q$ , which is available in current numerical studies, is limited because the momenta on the lattice are quantized and, in order to avoid discretization errors,  $|q| \ll 1$  ( $|q|a \ll 1$ , where  $a$  is the lattice spacing, if we do not work in lattice units). In order to enlarge the range of values of  $s_q$ , it is convenient to have a  $D$ -meson with non-zero momentum, corresponding to  $\vec{q}_1 \neq \vec{q}_2$ .
- To derive the properties of the resonance, it is not necessary to use the four-point correlation function. We can instead study the three-point  $H$ - $\Pi$ - $\Pi$  correlation  $G(t_1, t_2, \vec{q}_1, -\vec{q}_2)$  as a function of the momenta. There are several advantages in doing so. First of all, an additional information on the parameters of the resonance can be exploited when analysing the four-point correlation function. Moreover the signal of the  $H$ - $\Pi$ - $\Pi$  correlation is expected to be much less noisy than in the four-point case. A final advantage is that, by varying the quantum numbers of the field  $H(0)$  we can excite and study different resonances, for example a vector-like one.

We now present an example to show how  $R^{\mathcal{H}w}(t_2, \vec{q})$  would appear as a function of  $t_2$  for specific values of the mass and width of the resonance. What happens is the following. In the absence of final-state interactions,  $R^{\mathcal{H}w}(t_2, \vec{q})$  would display a plateau as a function of  $t_2$ . In the presence of the resonance, the second term in eq. (42) increases (decreases) exponentially in  $t_2$  for states with  $E_i < 2E_{\vec{q}}$  ( $E_i > 2E_{\vec{q}}$ ). It can easily be seen that only the states with  $E_i < 2E_{\vec{q}}$  give an appreciable contribution: the last term on the r.h.s. of eq. (42) gives a dramatic effect on the plateau for large values of  $t_2$ . This is shown in fig. 1, where the following parameters have been chosen:

1.  $L = 24$  and  $T = 64$ ;

2.  $\beta = 6.2$ , corresponding to  $a^{-1} \sim 2.9$  GeV;
3. a resonance with a mass  $M_\sigma = 1.9$  GeV and a width  $\Gamma(M_\sigma^2) = 0.3$  GeV, as was used for  $D$ -decays in ref. [5];
4.  $\Gamma(s) = \Gamma(M_\sigma^2) \sqrt{[(s - 4M_\pi^2)M_\sigma^2]/[(M_\sigma^2 - 4M_\pi^2)s]}$ ;
5. the masses of the pseudoscalar mesons have been taken to be  $M_\pi = 0.4$  GeV;
6. the external momenta are  $\vec{q}_1 = (1, 1, 0)$  and  $\vec{q}_2 = (-1, -1, 0)$ , so that there are only two intermediate states, of zero total momentum, corresponding to  $k = 0$  and  $k = 2\pi/L$ , and energy less than the energy of the external state; of the two states, the one with the two pions at rest does not contribute since  $\delta(4M_\pi^2) = 0$ .
7.  $A^W(s)$  has been taken to be constant and equal to 1.

By fitting the curve of fig. 1 as a function of  $t_2$  it is possible to extract  $A(s_q) \cos(s_q)$ , with  $s_q = 4[M_\pi^2 + 2(2\pi/La)^2]$ , and  $A(s_k) \sin(s_k)$ , with  $s_k = 4[M_\pi^2 + (2\pi/La)^2]$ . In the same way, by increasing the energy of the external two-pion state, and using also states with non-zero momentum, we can extract  $A_c$  and  $A_s$  as a function of  $s_q$  and extrapolate the amplitude and the phase to their physical value.

## Conclusion

We have shown that, in spite of the MTNGT, it is possible to extract the relevant information on two-body decay matrix elements, under the hypotheses that final-state interactions are dominated by nearby resonances and that the couplings are smooth in the momenta. We have also outlined the strategy to extract this information from the Euclidean correlation functions that can be computed in standard numerical simulations. The feasibility of our proposal requires a dedicated study, which will be presented elsewhere.

## Acknowledgements

We warmly thank M. Testa for many useful discussions. G.M. thanks the Theory Division of CERN for the kind hospitality during the completion of this work. We acknowledge the partial support by M.U.R.S.T., Italy.

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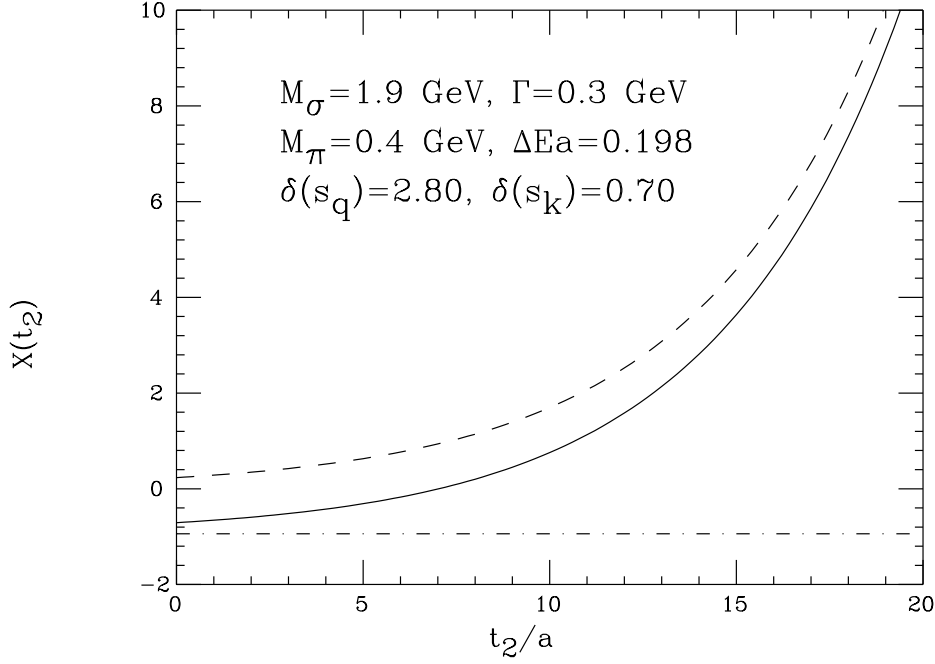


Figure 1:  $X(t_2) = R^{\mathcal{H}W}(t_2, \vec{q})$  as a function of  $t_2$  with the parameters given in the text.  $\Delta E a = 2(E_{\vec{q}} - E_{\vec{k}})a$ , where  $2E_{\vec{q}} = \sqrt{s_q} = 2\sqrt{M_\pi^2 + 2(2\pi/La)^2}$  and  $2E_{\vec{k}} = \sqrt{s_k} = 2\sqrt{M_\pi^2 + (2\pi/La)^2}$ .  $\delta(s_q)$  and  $\delta(s_k)$  are the corresponding phases. The dot-dashed line is the constant term corresponding to  $A^W(s_q) \cos \delta(s_q)$ , the dashed curve is the contribution of  $A^W(s_k) \sin \delta(s_k)$  and the solid curve is the sum of the two terms.